Déduction naturelle

OPTION INFORMATIQUE - TP n° 4.4 - Olivier Reynet

À la fin de ce chapitre, je sais :

- ☞ lire un séquent
- ☞ décrire les règles d'introduction et d'élimination
- ☞ justifier les principaux raisonnements de la logique classique
- ☞ construire un arbre de preuve démontrant une formule simple

A Utilisation des règles d'inférence

Prouver les séquents suivants :

A1. $\vdash p \rightarrow p$

Solution :

$$\dfrac{\dfrac{}{p \vdash p}\,\text{ax}}{\vdash p \rightarrow p}\,\rightarrow_i$$

A2. $p, \neg p \vdash \bot$

Solution :

$$\dfrac{\dfrac{}{p, \neg p \vdash p}\,\text{ax} \qquad \dfrac{}{p, \neg p \vdash \neg p}\,\text{ax}}{p, \neg p \vdash \bot}\,\neg_e$$

A3. $p, q \vdash p \wedge q$

Solution :

$$\dfrac{\dfrac{}{p, q \vdash p}\,\text{ax} \qquad \dfrac{}{p, q \vdash q}\,\text{ax}}{p, q \vdash p \wedge q}\,\wedge_i$$

A4. $p \wedge q \vdash q \wedge p$

Solution :

$$\dfrac{\dfrac{\dfrac{}{p\land q\vdash p\land q}\,\text{ax}}{p\land q\vdash q}\,\land_e \qquad \dfrac{\dfrac{}{p\land q\vdash p\land q}\,\text{ax}}{p\land q\vdash p}\,\land_e}{p\land q\vdash q\land p}\,\land_i$$

A5. $p\lor q\vdash q\lor p$

Solution :

$$\dfrac{\dfrac{}{p\lor q\vdash p\lor q}\,\text{ax} \qquad \dfrac{\dfrac{}{p\lor q, p\vdash p}\,\text{ax}}{p\lor q, p\vdash q\lor p}\,\lor_i \qquad \dfrac{\dfrac{}{p\lor q, q\vdash q}\,\text{ax}}{p\lor q, q\vdash q\lor p}\,\lor_i}{p\lor q\vdash q\lor p}\,\lor_e$$

A6. $q\vdash p\to q$

Solution :

$$\dfrac{\dfrac{}{q, p\vdash q}\,\text{ax}}{q\vdash p\to q}\,\to_i$$

A7. $p\land q\vdash p\to q$

Solution :

$$\dfrac{\dfrac{\dfrac{}{p\land q, p\vdash p\land q}\,\text{ax}}{p\land q, p\vdash q}\,\land_e}{p\land q\vdash p\to q}\,\to_i$$

A8. $p, q\land r\vdash p\land q$

Solution :

$$\dfrac{\dfrac{}{p, q\land r\vdash p}\,\text{ax} \qquad \dfrac{\dfrac{}{p, q\land r\vdash q\land r}\,\text{ax}}{p, q\land r\vdash q}\,\land_e}{p, q\land r\vdash p\land q}\,\land_i$$

A9. $p\land q, r\land s\vdash p\land s$

Solution :

$$\dfrac{\dfrac{\dfrac{}{p\land q, r\land s\vdash p\land q}\,\text{ax}}{p\land q, r\land s\vdash p}\,\land_e \qquad \dfrac{\dfrac{}{p\land q, r\land s\vdash r\land s}\,\text{ax}}{p\land q, r\land s\vdash s}\,\land_e}{p\land q, r\land s\vdash p\land s}\,\land_i$$

A10. $a \to \neg a \vdash \neg a$

Solution :

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a \to \neg a, a \vdash a \to \neg a}\,\text{ax} \qquad \dfrac{}{a \to \neg a, a \vdash a}\,\text{ax}}{a \to \neg a, a \vdash \neg a}\,\to_e \qquad \dfrac{}{a \to \neg a, a \vdash a}\,\text{ax}}{a \to \neg a, a \vdash \bot}\,\neg_e}{a \to \neg a \vdash \neg a}\,\neg_i$$

B Preuves intermédiaires

Prouver les séquents suivants :

B1. $p \to q \vdash \neg q \to \neg p$

Solution : On pose $\Gamma = p \to q, \neg q, p$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash p}\,\text{ax} \qquad \dfrac{}{\Gamma \vdash p \to q}\,\text{ax}}{\Gamma \vdash q}\,\to_e \qquad \dfrac{}{\Gamma \vdash \neg q}\,\text{ax}}{\Gamma \vdash \bot}\,\neg_e}{p \to q, \neg q \vdash \neg p}\,\neg_i}{p \to q \vdash \neg q \to \neg p}\,\to_i$$

B2. $\neg a \vee b \vdash a \to b$

Solution :

$$\dfrac{\dfrac{\dfrac{}{\neg a \vee b, a \vdash \neg a \vee b}\,\text{ax} \qquad \dfrac{\dfrac{\dfrac{}{\neg a \vee b, a, \neg a \vdash a}\,\text{ax} \qquad \dfrac{}{\neg a \vee b, a, \neg a \vdash \neg a}\,\text{ax}}{\neg a \vee b, a, \neg a \vdash \bot}\,\neg_e}{\neg a \vee b, a, \neg a \vdash b}\,\bot_e \qquad \dfrac{}{\neg a \vee b, a, b \vdash b}\,\text{ax}}{\neg a \vee b, a \vdash b}\,\vee_e}{\neg a \vee b \vdash a \to b}\,\to_i$$

B3. $a \to b \vdash \neg a \vee b$

Solution :

$$\dfrac{\dfrac{}{a \to b \vdash \neg a \vee a}\,\text{te} \qquad \dfrac{\dfrac{}{a \to b, \neg a \vdash \neg a}\,\text{ax}}{a \to b, \neg a \vdash \neg a \vee b}\,\vee_i \qquad \dfrac{\dfrac{\dfrac{}{a \to b, a \vdash a \to b}\,\text{ax} \qquad \dfrac{}{a \to b, a \vdash a}\,\text{ax}}{a \to b, a \vdash b}\,\to_e}{a \to b, a \vdash \neg a \vee b}\,\vee_i}{a \to b \vdash \neg a \vee b}\,\vee_e$$

B4. $a \to (b \to c) \vdash (a \wedge b) \to c$

Solution :

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$a \to (b \to c), a \wedge b \vdash a \wedge b$}
\RightLabel{$\wedge_e$}
\UnaryInfC{$a \to (b \to c), a \wedge b \vdash b$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$a \to (b \to c), a \wedge b \vdash a \wedge b$}
\RightLabel{$\wedge_e$}
\UnaryInfC{$a \to (b \to c), a \wedge b \vdash a$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$a \to (b \to c), a \wedge b \vdash a \to (b \to c)$}
\RightLabel{$\to_e$}
\BinaryInfC{$a \to (b \to c), a \wedge b \vdash b \to c$}
\RightLabel{$\to_e$}
\BinaryInfC{$a \to (b \to c), a \wedge b \vdash c$}
\RightLabel{$\to_i$}
\UnaryInfC{$a \to (b \to c) \vdash (a \wedge b) \to c$}
\end{prooftree}
$$

B5. $(a \wedge b) \to c \vdash a \to (b \to c)$

Solution : On pose $\Gamma = (a \wedge b) \to c, a, b$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash a$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash b$}
\RightLabel{$\wedge_i$}
\BinaryInfC{$\Gamma \vdash a \wedge b$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash (a \wedge b) \to c$}
\RightLabel{$\to_e$}
\BinaryInfC{$(a \wedge b) \to c, a, b \vdash c$}
\RightLabel{$\to_i$}
\UnaryInfC{$(a \wedge b) \to c, a \vdash b \to c$}
\RightLabel{$\to_i$}
\UnaryInfC{$(a \wedge b) \to c \vdash a \to (b \to c)$}
\end{prooftree}
$$

B6. $a \to (b \to c), b \to a \vdash b \to c$

Solution : On pose $\Gamma = a \to (b \to c), b \to a, b$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash b$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash b \to a$}
\RightLabel{$\to_e$}
\BinaryInfC{$\Gamma \vdash a$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash a \to (b \to c)$}
\RightLabel{$\to_e$}
\BinaryInfC{$\Gamma \vdash b \to c$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash b$}
\RightLabel{$\to_e$}
\BinaryInfC{$\Gamma \vdash c$}
\RightLabel{$\to_i$}
\UnaryInfC{$a \to (b \to c), b \to a \vdash b \to c$}
\end{prooftree}
$$

B7. $p \to (q \vee r), \neg q, \neg r \vdash \neg p$

Solution : On pose $\Gamma = p \to (q \vee r), \neg q, \neg r$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, p \vdash p \to (q \vee r)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, p \vdash p$}
\RightLabel{$\to_e$}
\BinaryInfC{$\Gamma, p \vdash q \vee r$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, p, q \vdash q$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, p, q \vdash \neg q$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma, p, q \vdash \bot$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, p, r \vdash r$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, p, r \vdash \neg r$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma, p, r \vdash \bot$}
\RightLabel{$\vee_e$}
\TrinaryInfC{$\Gamma, p \vdash \bot$}
\RightLabel{$\neg_i$}
\UnaryInfC{$p \to (q \vee r), \neg q, \neg r \vdash \neg p$}
\end{prooftree}
$$

B8. $p \to (q \to r), p, \neg r \vdash \neg q$

Solution : On pose $\Gamma = p \to (q \to r), p, \neg r$.

$$\dfrac{\dfrac{\dfrac{}{\Gamma, q \vdash \neg r}\,\text{ax} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, q \vdash p}\,\text{ax} \quad \dfrac{}{\Gamma, q \vdash p \to (q \to r)}\,\text{ax}}{\Gamma, q \vdash q \to r}\,\to_e \quad \dfrac{}{\Gamma, q \vdash q}\,\text{ax}}{\Gamma, q \vdash r}\,\to_e}{\Gamma, q \vdash \bot}\,\neg_e}{p \to (q \to r), p, \neg r \vdash \neg q}\,\neg_i$$

C Preuves plus complexes

Prouver les séquents suivants :

C1. $q \to r, \neg q \to \neg p \vdash p \to r$

Solution : On pose $\Gamma = q \to r, \neg q \to \neg p, p$.

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash q \neg q}\,\text{te} \quad \dfrac{\dfrac{}{\Gamma, q \vdash q \to r}\,\text{ax} \quad \dfrac{}{\Gamma, q \vdash q}\,\text{ax}}{\Gamma, q \vdash r}\,\to_e \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \neg q \vdash \neg q \to \neg p}\,\text{ax} \quad \dfrac{}{\Gamma, \neg q \vdash \neg q}\,\text{ax}}{\Gamma, \neg q \vdash \neg p}\,\to_e \quad \dfrac{}{\Gamma, \neg q \vdash p}\,\text{ax}}{\Gamma, \neg q \vdash \bot}\,\neg_e}{\Gamma, \neg q \vdash r}\,\bot_e}{}}{\Gamma \vdash r}\,\vee_e}{q \to r, \neg q \to \neg p \vdash p \to r}\,\to_i$$

C2. $(p \wedge q) \to r \vdash (p \to r) \vee (q \to r)$

Solution : On pose $\Gamma = (p \wedge q) \to r$ et $\psi = (p \to r) \vee (q \to r)$

$$\dfrac{\dfrac{}{\Gamma \vdash p \neg p}\,\text{te} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, p, q \vdash \Gamma}\,\text{ax} \quad \dfrac{\dfrac{}{\Gamma, p, q \vdash p}\,\text{ax} \quad \dfrac{}{\Gamma, p, q \vdash q}\,\text{ax}}{\Gamma, p, q \vdash p \wedge q}\,\wedge_i}{\Gamma, p, q \vdash r}}{\Gamma, p \vdash q \to r}\,\to_i}{\Gamma, p \vdash \psi}\,\vee_i \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \neg p, p \vdash p}\,\text{ax} \quad \dfrac{}{\Gamma, \neg p, p \vdash \neg p}\,\text{ax}}{\Gamma, \neg p, p \vdash \bot}\,\neg_e}{\Gamma, \neg p, p \vdash r}\,\bot_e}{\Gamma, \neg p \vdash p \to r}\,\to_i}{\Gamma, \neg p \vdash \psi}\,\vee_i}{\Gamma \vdash \psi}\,\vee_e$$

C3. $\neg(a \vee b) \vdash \neg a \wedge \neg b$ (Loi de De Morgan)

Solution :

$$\dfrac{\dfrac{}{\neg(a \vee b) \vdash \neg a}\,\text{à prouver} \qquad \dfrac{}{\neg(a \vee b) \vdash \neg b}\,\text{à prouver}}{\neg(a \vee b) \vdash \neg a \wedge \neg b}\,\wedge_i$$

On a divisé l'objectif en deux sous-objectifs et les preuves seront identiques pour les deux.

$$\dfrac{\dfrac{}{\neg(a\vee b)\vdash \neg a\vee a}\,\text{te} \qquad \dfrac{}{\neg(a\vee b),\neg a\vdash\neg a}\,\text{ax} \qquad \dfrac{}{\neg(a\vee b), a\vdash \neg a}\,\text{à prouver}}{\neg(a\vee b)\vdash\neg a}\,\vee_e$$

On cherche maintenant à prouver $\neg(a\vee b), a\vdash\neg a$.

$$\dfrac{\dfrac{\dfrac{}{\neg(a\vee b),a\vdash\neg(a\vee b)}\,\text{ax} \qquad \dfrac{\dfrac{}{\neg(a\vee b),a\vdash a}\,ax}{\neg(a\vee b),a\vdash a\vee b}\,\vee_i}{\neg(a\vee b),a\vdash\bot}\,\neg_e}{\neg(a\vee b),a\vdash\neg a}\,\bot_e$$

On procède de même pour $\neg(a\vee b)\vdash\neg b$. L'utilisation du tiers-exclu fait de cette preuve un exemple de logique classique. On peut procéder sans et en faire ainsi un exemple de preuve en logique intuitionniste.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(a\vee b),a\vdash\neg(a\vee b)}\,ax \qquad \dfrac{\dfrac{}{\neg(a\vee b),a\vdash a}\,ax}{\neg(a\vee b),a\vdash a\vee b}\,\vee_i}{\neg(a\vee b),a\vdash\bot}\,\neg_e}{\neg(a\vee b)\vdash\neg a}\,\neg_i \qquad \dfrac{\dfrac{\dfrac{}{\neg(a\vee b),b\vdash\neg(a\vee b)}\,ax \qquad \dfrac{\dfrac{}{\neg(a\vee b),b\vdash b}\,ax}{\neg(a\vee b),b\vdash a\vee b}\,\vee_i}{\neg(a\vee b),b\vdash\bot}\,\neg_e}{\neg(a\vee b)\vdash\neg b}\,\neg_i}{\neg(a\vee b)\vdash\neg a\wedge\neg b}\,\wedge_i$$